CSCC37F18 Midterm Questions

Question 1

[10 marks]

Let $x,\ y \in \mathbb{R}$. Recall that $fl(x),\ fl(y) \in \mathbb{R}_b(t,s)$ denote the floating-point representations of x and y, respectively, where $fl(x) = x(1-\delta_x)$, $fl(y) = y(1-\delta_y)$, and $\delta_x,\ \delta_y$ quantify the relative roundoff errors in the respective representations.

In lecture, we showed that a typical computer estimates the product of x and y as

$$fl(fl(x) \cdot fl(y)) = (x \cdot y)(1 - \delta_\cdot)$$

where $|\delta_\cdot| \le 3$ eps. Using similar techniques, derive a tight error bound for computer division.

Question 2

[15 marks]

Consider calculating the LU-factorization of $A \in \mathcal{R}^{5\times 5}$, using Gaussian Elimination with partial pivoting. After stage 4 of the elimination we have

$$\mathcal{L}_4\, \mathcal{P}_4\, \mathcal{L}_3\, \mathcal{P}_3\, \mathcal{L}_2\, \mathcal{P}_2\, \mathcal{L}_1\, \mathcal{P}_1\, A = U \tag{1}$$

where $\mathcal{P}_i$, $\mathcal{L}_i$ are, respectively, the permutation and Gauss transform used in the i-th stage of the elimination, and U is the upper-triangular factor of the factorization.

The final form of the factorization is

$$P\, A = L\, U$$

where

$$P = \mathcal{P}_4\, \mathcal{P}_3\, \mathcal{P}_2\, \mathcal{P}_1$$

and

$$L = \tilde{\mathcal{L}}_1^{-1}\, \tilde{\mathcal{L}}_2^{-1}\, \tilde{\mathcal{L}}_3^{-1}\, \mathcal{L}_4^{-1}$$

a. Express $\tilde{\mathcal{L}}_1^{-1}$ in terms of the original $\mathcal{P}_i$ and $\mathcal{L}_i$ appearing in (1). **Show all of your work.**

b. Given that

$$\mathcal{L}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 & 0 \\ m_{31} & 0 & 1 & 0 & 0 \\ m_{41} & 0 & 0 & 1 & 0 \\ m_{51} & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \mathcal{P}_1 &\equiv \mathcal{P}_{14} \\ \mathcal{P}_2 &\equiv \mathcal{P}_{25} \\ \mathcal{P}_3 &\equiv \mathcal{P}_{34} \\ \mathcal{P}_4 &\equiv \mathcal{P}_{45} \end{aligned}$$

($\mathcal{P}_{ij}$ interchanges rows i and j for $j > i$), and considering your answer in part (**a**), write out the matrix representation of $\tilde{\mathcal{L}}_1^{-1}$ showing precisely the sign and position of the four multipliers m_{i1}.

Question 3

[15 marks]

Consider the linear system $Ax = b$ where

$$A = \begin{bmatrix} 3 & 5 & 9 \\ 4 & 4 & 4 \\ 1 & 5 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 40 \\ 24 \\ 26 \end{bmatrix}.$$

a. Compute the $PA = LU$ factorization of A. Use exact arithmetic. Show all intermediate calculations, including Gauss transforms and permutation matrices.

b. Use the factorization computed in part (**a**) to solve the system.

c. Why is Gaussian Elimination usually implemented as in this question (i.e., $PA = LU$ is computed separately, and then the factorization is used to solve $Ax = b$)?

Question 4

[10 marks]

In lecture we saw that Gaussian elimination with partial pivoting usually, but not always, leads to a stable factorization of $A \in \mathcal{R}^{n\times n}$. A stable factorization is guaranteed if we use *full* pivoting, which employs both row and column interchanges before the k-th stage of the elimination to ensure that the largest element in magnitude in the $(n-k) \times (n-k)$ submatrix finds its way to the pivot position.

Full pivoting leads to a $PAQ = LU$ factorization, where P and Q are permutation matrices. Show how this factorization can be used to solve $Ax = b$.

Question 5

[10 marks]

Explain exactly how row pivoting controls the growth of roundoff error during the LU factorization process. Use a small example and picture as was done in lecture.