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1. $(0.1)_{10} = (0.0\overline{0011})_2$

$= (0.\overline{0022})_3$

$= (0.0\overline{12})_4$

$= (0.0\overline{2})_5$

$= (0.0\overline{3})_6$

$= (0.\overline{0462})_7$

$= (0.0\overline{6314})_8$

$= (0.\overline{08})_9$

Convert $(0.1)_{10}$ to base i for $i = 2$ to 9

$\therefore$ cant be done

2. Clearly if a whole number can be represented in binary with a finite expression then it also can in decimal. So only look at binary fractions.

If x is finitely represented in binary

then $x = \sum_{i=1}^{n} a_i \left(\frac{1}{2^i}\right)$ where $a_i = 0$ or 1

and we know $\left(\frac{1}{2^i}\right)$ can be finitely represented in decimal ex: $\frac{1}{2} = 0.5$, $\frac{1}{4} = 0.25$, $\frac{1}{8} = 0.125$, ...

No binary fraction cant be finitely represented in decimal

3. a) $x \approx 0$ ← subtraction outside radical

$x \approx \beta$ ← subtraction inside radical

↳ not a problem since the contribution of the radical is insignificant to the overall result

$(f(x) \approx \beta,\ \beta >> 0)$

b) $$\lim_{x \to 0} \text{cond}(f(x)) = \lim_{x \to 0} \left| \frac{x f'(x)}{f(x)} \right|$$

$$= \lim_{x \to 0} \left| \frac{x \cdot \frac{x}{\sqrt{\beta^2 - x^2}}}{\beta - \sqrt{\beta^2 - x^2}} \right|$$

$$= \lim_{x \to 0} \left| \frac{x^2}{\beta\sqrt{\beta^2 - x^2} - \beta^2 + x^2} \right|$$

$$= \frac{0}{\beta\sqrt{\beta^2 - 0^2} - \beta^2 + 0^2} = \frac{0}{\beta^2 - \beta^2} = \frac{0}{0}$$

indeterminate form

use l'hopital's

$$= \lim_{x \to 0} \left| \frac{x f''(x) + 1 \cdot f'(x)}{f'(x)} \right|$$

$$= \lim_{x \to 0} \left| \frac{\dfrac{x\beta^2}{(\beta^2 - x^2)^{3/2}} + \dfrac{x}{\sqrt{\beta^2 - x^2}}}{\dfrac{x}{\sqrt{\beta^2 - x^2}}} \right|$$

$$= \lim_{x \to 0} \left| \frac{\dfrac{x\beta^2}{(\beta^2 - x^2)^{3/2}}}{\dfrac{x}{\sqrt{\beta^2 - x^2}}} + 1 \right|$$

$$= \lim_{x \to 0} \left| \frac{\beta^2}{\beta^2 - x^2} + 1 \right|$$

$$= \left| \frac{\beta^2}{\beta^2} + 1 \right|$$

$$= 1 + 1 \quad = 2$$

$\therefore \lim_{x \to 0} \operatorname{cond}(f(x)) = 2$, which in the context of a single function evaluation is extremely good. so it seems subtractive cancellation might not be a problem when $x \approx 0$

c) $f(x) = \beta - \sqrt{\beta^2 - x^2}$

$$= \beta - \sqrt{\beta^2 - x^2} \cdot \frac{\beta + \sqrt{\beta^2 - x^2}}{\beta + \sqrt{\beta^2 - x^2}}$$

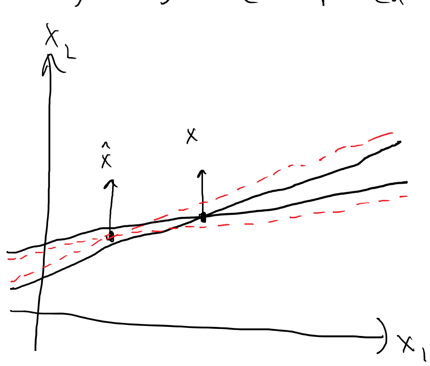
$$= \frac{x^2}{\beta + \sqrt{\beta^2 - x^2}}$$

can be used when $x \approx 0$ since no harmful subtractive cancellation in this range

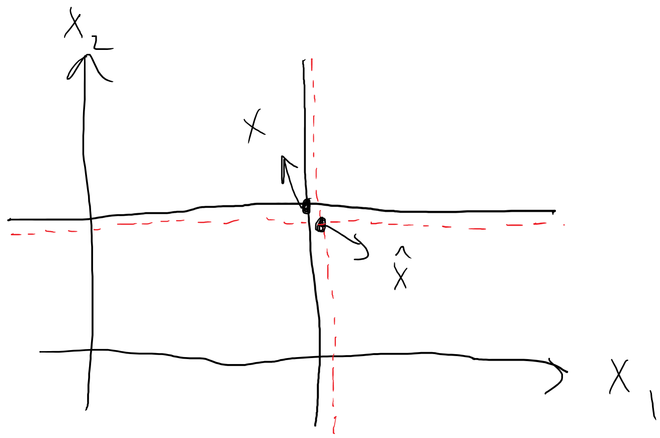
4. see week 9 and 10 notes

5. a) $Ax = b$ true system, x true solution

$(A+E)\hat{x} = b$ computed system, $\hat{x}$ computed solution

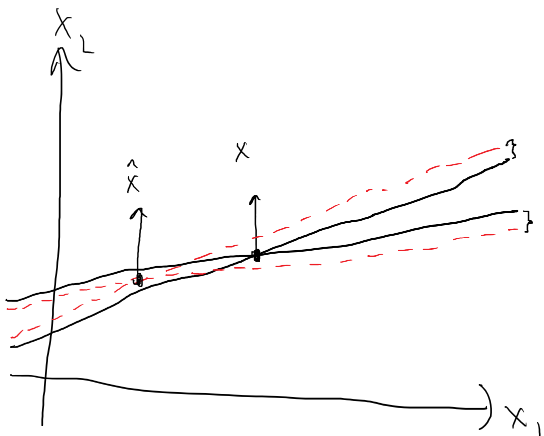


poorly conditioned



perfectly conditioned

b)



residual manifests as the orthogonal distance between solid and dashed lines measured at x_1 and x_2

c) 3 planes all mutually orthogonal to each other intersecting at a single point

6. a) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{2\varepsilon}\begin{bmatrix} 2+\varepsilon & -4 \\ -1 & 2 \end{bmatrix}$$

$\|A\|_1$ = max abs col sum

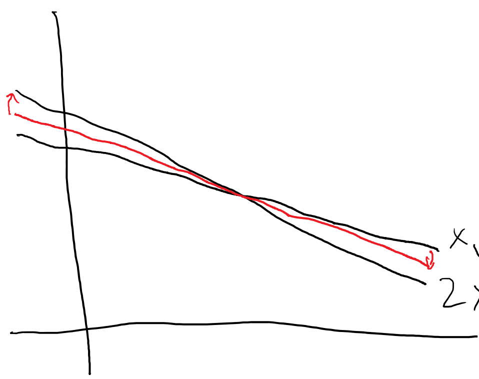
$$\|A\|_1 = 6 + \varepsilon$$

$$\|A^{-1}\|_1 = \frac{3}{\varepsilon}$$

$$\text{cond}_1(A) = \|A\|_1 \|A^{-1}\|_1 = \frac{18 + 3\varepsilon}{\varepsilon}$$

$$\lim_{\varepsilon \to 0} \text{cond}_1(A) = \infty$$

b)



as $\varepsilon \to 0$, lines become increasingly more parallel and effect on POI is more dramatic

7. $\hat{X}_{i+1} = \hat{X}_i + Z_i$

$$= \hat{X}_i + (A+E)^{-1} r_i$$

[when solving $Az_i = r_i$, computer is solving $(A+E)\hat{z}_i = r_i \Rightarrow \hat{z}_i = (A+E)^{-1} r_i$]

$$= \hat{X}_i + (A+E)^{-1}(b - A\hat{X}_i)$$

$$\Rightarrow \hat{x}_{i+1} - \hat{x}_i = (A+E)^{-1}(b - A\hat{x}_i)$$

$$\Rightarrow (A+E)(\hat{x}_{i+1} - \hat{x}_i) + A\hat{x}_i = b$$

$$\Rightarrow (A+E)\hat{x}_{i+1} - E\hat{x}_i = b \quad (1)$$

We have $Ax = b \Rightarrow Ax + Ex - Ex = b$

$$\Rightarrow (A+E)x - Ex = b \quad (2)$$

(1) − (2) :

$$(A+E)\hat{x}_{i+1} - (A+E)x - E\hat{x}_i + Ex = b - b$$

$$(A+E)(\hat{x}_{i+1} - x) - E(\hat{x}_i - x) = 0$$

$$(A+E)(\hat{x}_{i+1} - x) = E(\hat{x}_i - x)$$

$$x_{i+1} - x = (A+E)^{-1} E(\hat{x}_i - x)$$

$$\|x_{i+1} - x\| = \|(A+E)^{-1}E(\hat{x}_i - x)\|$$

$$\|x_{i+1} - x\| \leq \|(A+E)^{-1}E\| \, \|\hat{x}_i - x\|$$

Hence, convergence if $\|(A+E)^{-1}E\| < 1$

since $\|E\| \leq k \cdot eps \cdot \|A\|$, $\|(A+E)^{-1}\| \approx \|A^{-1}\|$

if A is not poorly conditioned, $\|A^{-1}\|$ not large, and since $\|E\|$ small

$$\|(A+E)^{-1}E\| \leq \|(A+E)^{-1}\|\|E\| \approx \|A^{-1}\|\|E\| < 1$$

another way:

$$\hat{x}_{i+1} = \hat{x}_i + (A+E)^{-1}(b - A\hat{x}_i)$$

$$= \hat{x}_i + (A+E)^{-1}(Ax - A\hat{x}_i)$$

$$= \hat{x}_i + (A+E)^{-1}A(x - \hat{x}_i)$$

$$\hat{x}_{i+1} - x = \hat{x}_i - x + (A+E)^{-1}A(x - \hat{x}_i)$$

$$= \left(I - (A+E)^{-1}A\right)(\hat{x}_i - x)$$

$$\|\hat{x}_{i+1} - x\| \leq \underbrace{\|I - (A+E)^{-1}A\|}_{<1} \|\hat{x}_i - x\|$$

$$I - (A+E)^{-1}A = I - (A+E)^{-1}(A+E-E)$$

$$= I - (A+E)^{-1}(A+E) + (A+E)^{-1}E$$

$$= I - I + (A+E)^{-1}E$$

$$= (A+E)^{-1}E$$