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https://snoopysnipe.github.io/ta/c37f20/

Approximation / Interpolation

3 techniques for computing interpolating polynomial

1. Method of undetermined coefficients
2. Lagrange basis
3. Newton (divided-differences) basis

ex: compute the quadratic polynomial interpolating $\{(0,3), (1,7), (2,37)\}$

1. $p(x) = \sum_{i=0}^{n} a_i x^i$

we have 3 data points, $n=2$

$p(x_0) = y_0$
$p(x_1) = y_1$ Solve for a_i
$p(x_2) = y_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 37 \end{bmatrix}$$

Use $PV = LU$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Ld = P_y \Rightarrow d = \begin{bmatrix} 3 \\ 34 \\ -13 \end{bmatrix}$$

$$Ua = d \Rightarrow a = \begin{bmatrix} 3 \\ -9 \\ 13 \end{bmatrix}$$

$$\therefore\ p(x) = 3 - 9x + 13x^2$$

<u>check:</u> $p(0) = 3$

$p(1) = 3 - 9 + 13 = 7$ ✓

$p(2) = 3 - 18 + 52 = 37$

② $p(x) = \sum_{i=0}^{n} l_i(x) y_i$

where $l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \dfrac{(x - x_j)}{(x_i - x_j)}$

$$l_0(x) = \frac{(x-1)(x-2)}{2}$$

$$l_1(x) = -x(x-2)$$

$$l_2(x) = \frac{x(x-1)}{2}$$

$\therefore p(x) = \dfrac{3(x-1)(x-2)}{2} - 7x(x-2) + \dfrac{37x(x-1)}{2}$

Check: $p(0) = 3 = y_0$ ✓

$p(1) = 7 = y_1$

$p(2) = 37 = y_2$

$$= \frac{3}{2}(x^2 - 3x + 2) - 7x^2 + 14x + \frac{37}{2}(x^2 - x)$$

$$= \frac{3}{2}x^2 - \frac{9}{2}x + 3 - 7x^2 + 14x + \frac{37}{2}x^2 - \frac{37}{2}x$$

$$= 3 - 9x + 13x^2$$

3) $p(x) = \sum_{i=0}^{n} \left[a_i \prod_{j=0}^{i-1} (x - x_j) \right]$

where a_i is the i^{th} divided difference on $[x_0, x_1, \dots, x_i]$

$$p(x) = a_0 + (x - x_0)a_1 + (x - x_0)(x - x_1)a_2$$

where $a_0 = y[x_0] = y_0$

$$a_1 = y[x_1, x_0] = \frac{y_1 - y_0}{x_1 - x_0}$$

$$a_2 = y[x_2, x_1, x_0]$$

$$= \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\dfrac{y_2 - y_1}{x_2 - x_1} - \dfrac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

x_i	$y[x_i]$	$y[x_{i+1}, x_i]$	$y[x_{i+2}, x_{i+1}, x_i]$
0	3		
		$\frac{7-3}{1-0} = 4$	
1	7		$\frac{30-4}{2-0} = 13$
		$\frac{37-7}{2-1} = 30$	
2	37		

$a_0 = 3, \ a_1 = 4, \ a_2 = 13$

$\therefore p(x) = 3 + 4x + 13x(x-1)$

<u>Check:</u> $p(0) = 3$

$p(1) = 7$

$p(2) = 3 + 8 + 26 = 37$ ✓

$= 3 + 4x + 13x^2 - 13x$

$= 3 - 9x + 13x^2$

Relative Efficiency of the 3 methods

① - requires solving $(n+1) \times (n+1)$ linear system to compute coefficients of interpolating polynomial
- evaluation of resulting polynomial is reasonable if rewritten in nested form (Horner's Rule).
- Vandermonde matrix is usually poorly conditioned if n is large

② - Constructing Lagrange basis polynomial is essentially free. The i^{th} coefficient is just y_i for the i^{th} data point.

- Evaluation is expensive at non-interpolating points.

Mainly used only in proofs of theorems. Not very practical.

③ - Coefficients are not expensive to compute with divided difference table and resulting polynomial is easily written in nested form for efficient evaluation

Which method is best if we need to add data points? ie: add (3, 141) to previous example

① Add column and row to V, refactor matrix, condition is potentially worsened

② Construction still free. But each basis function changes and evaluation becomes even more expensive.

③ Just add row and column to divided differences table, but most calculations are reused

x_i	$y[x_i]$	$y[x_{i+1}, x_i]$	$y[x_{i+2}, x_{i+1}, x_i]$	$y[x_{i+3}, x_{i+2}, x_{i+1}, x_i]$
0	3			
		$\frac{7-3}{1-0} = 4$		
1	7		$\frac{30-4}{2-0} = 13$	
		$\frac{37-7}{2-1} = 30$		$\frac{37-13}{3-0} = 8$
2	37		$\frac{104-30}{3-1} = 37$	
		$\frac{141-37}{3-2} = 104$		
3	141			

$a_0 = 3$, $a_1 = 4$, $a_2 = 13$, $a_3 = 8$

note: a_0, a_1, a_2 remain unchanged

$$p(x) = 3 + 4x + 13x(x-1) + 8x(x-1)(x-2)$$

note: first 3 terms remain unchaged

Check: $p(0) = 3$

$p(1) = 7$

$p(2) = 37$

$p(3) = 3 + 12 + 78 + 48 = 141$ ✓