

- ① Conditioning of functions (derive the expression with Taylor Series, but there are many other methods)
- ② Matrix and vector norms

$$1) f(x) = f(\hat{x} + x - \hat{x}) \\ = f(\hat{x}) + (x - \hat{x})f'(\hat{x}) + \frac{(x - \hat{x})^2}{2!} f''(\xi) \quad \xi \in [\hat{x}, x]$$

$$\Rightarrow f(x) \approx f(\hat{x}) + (x - \hat{x})f'(\hat{x})$$

Taylor series of f centered around \hat{x}
 where $\hat{x} = f(x) = x(1 - \delta_x)$

The approximation is good when \hat{x} is near x

$$\Rightarrow |f(x) - f(\hat{x})| \approx |f'(\hat{x})| |x - \hat{x}|$$

$$\Rightarrow \frac{|f(x) - f(\hat{x})|}{|f(x)|} \approx \frac{|x f'(x)|}{|f(x)|} \frac{|x - \hat{x}|}{|x|}$$

∴ The relative error in $f(\hat{x})$ is proportional to the relative error in \hat{x} with proportionality factor

$$\text{cond}(f) = \frac{|x f'(\hat{x})|}{|f(x)|}$$

in context of measuring condition of functions,
 \hat{x} is usually sufficiently close to x ,
 allowing us to use the approximation

$$\text{cond}(f) \approx \frac{|x f'(x)|}{|f(x)|} \leftarrow \text{notice it is } f'(x) \text{ NOT } f'(\hat{x})$$

ex: $f(x) = \sqrt{x}$

$$\text{cond}(f) = \frac{|x \frac{1}{2\sqrt{x}}|}{|\sqrt{x}|}$$

$$= \frac{|x|}{|2\sqrt{x} \sqrt{x}|}$$

$$= \frac{|x|}{|2\hat{x} \sqrt{x}|}$$

$$= \frac{1}{2} \quad \text{well-conditioned}$$

ex: $f(x) = \frac{10}{1-x^2}$

$$\text{cond}(f) = \frac{|x \frac{20\hat{x}}{(1-\hat{x}^2)^2}|}{|\frac{10}{1-x^2}|}$$

$$= \frac{|2x^2|}{|(1-x^2)|}$$

poorly conditioned
 when $|x| \approx 1$

2) Vector Norms $\bar{x} \in \mathbb{R}^n$

$$\|\bar{x}\|_1 = \sum_{i=1}^n |x_i|$$

$$\|\bar{x}\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}}$$

$$\|\bar{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

in general, for $p > 0$

$$\|\bar{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Properties

1) $\|\bar{x}\| > 0$ if $\bar{x} \neq \bar{0}$

2) $\| \gamma \bar{x} \| = |\gamma| \|\bar{x}\|$ for any scalar γ

3) $\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$ triangle inequality

also,

$$\|\bar{x}\|_1 \geq \|\bar{x}\|_2 \geq \|\bar{x}\|_\infty$$

$$\|\bar{x}\|_1 \leq \sqrt{n} \|\bar{x}\|_2$$

$$\|\bar{x}\|_2 \leq \sqrt{n} \|\bar{x}\|_\infty$$

$$\|\bar{x}\|_1 \leq n \|\bar{x}\|_\infty$$

ex! $\bar{x} = [3, 5, -7, 8]$

$$\|\bar{x}\|_1 = 23$$

$$\|\bar{x}\|_2 = \sqrt{147}$$

$$\|\bar{x}\|_\infty = 8$$

Matrix Norms $A \in \mathbb{R}^{n \times m}$

$$\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}| = \text{max absolute column sum}$$

$$\|A\|_\infty = \max_i \sum_{j=1}^m |a_{ij}| = \text{max absolute row sum}$$

Properties

1) $\|A\| > 0$ if $A \neq 0$

2) $\|\gamma A\| = |\gamma| \|A\|$ for any scalar γ

3) $\|A+B\| \leq \|A\| + \|B\|$

4) $\|AB\| \leq \|A\| \|B\|$

5) $\|Ax\| \leq \|A\| \|x\|$ for any vector x

ex: $A = \begin{bmatrix} 1 & -7 \\ -2 & -3 \end{bmatrix}$

$$\|A\|_1 = 10$$

$$\|A\|_\infty = 8$$

Additional Notes:

$$\text{In general, } \|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Also, $\|A\|_2 = \sigma_{\max}(A)$; the largest singular value of matrix A

$$\|A\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^m |a_{ij}|^2}$$
 ; the Frobenius norm