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https://snoopysnipe.github.io/ta/c37f20/

Gaussian Elimination without pivoting

(1) Compute LU factorization of matrix $A \in \mathbb{R}^{n \times n}$

(2) Use the factorization to solve system of linear equations $Ax = b$

1) Reduce A to upper triangular form using $n-1$ Gauss transforms L_i

↳ L_i annihilates all elements on i-th column of A under element a_{ii}

$\Rightarrow$ We have $L_{n-1} L_{n-2} \dots L_2 L_1 A = U$

$\Rightarrow A = L_1^{-1} L_2^{-1} \dots L_{n-2}^{-1} L_{n-1}^{-1} U$

$\Rightarrow A = LU$

where $L = L_1^{-1} L_2^{-1} \dots L_{n-2}^{-1} L_{n-1}^{-1}$

Compute L by toggling the signs and combining the elements under the diagonal

→ to calculate individual inverses

→ matrix multiplication

2) Now that we have L and U, solve $Ax = b$ by:

$Ax = b \iff LUx = b$

i) solve $Ld = b$ for d (forward substitution)

ii) solve $Ux = d$ for x (backward substitution)

motivation: much easier to solve due to the triangular forms of L and U.

More efficient to solve

$Ax = b_1$

$Ax = b_2$

$Ax = b_3$

$\vdots$

Since we can reuse the LU factorization as opposed to computing

$x = A^{-1} b_1$

$x = A^{-1} b_2$

$x = A^{-1} b_3$

$\vdots$

General Algo (given A and b) to compute L_i:

for $i = 1, \ldots, n-1$

1. Compute L_i
   ↳ Identity matrix and elements under l_{ii} are $-\frac{a_{ij}}{a_{ii}}$ for $j > i$
2. $A \leftarrow L_i A$
   ↳ only a_{kj} $k>i, j>i$ need to be calculated and a_{ij} $j>i$ become 0
   ↳ use this new matrix for the next iteration

ex: solve $Ax=b$

$$A=\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -2 & 4 & 1 \end{bmatrix} \qquad b=\begin{bmatrix} 1 \\ 8 \\ 7 \end{bmatrix}$$

$$L_1=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad L_1A=\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$$L_2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \boxed{L_2L_1A=\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U}$$

We have $L_2L_1A=U \iff A=\underbrace{L_1^{-1}L_2^{-1}}_{L}U$

$$\boxed{L=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}}$$

We have $Ax=b \iff LUx=b$

Let $Ux=d$

Solve $Ld=b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 8 \\ 7 \end{bmatrix}$$

$d_1=1$

$\Rightarrow d_2=8-d_1=8-1=7$

$\Rightarrow d_3=7+d_1-d_2=7+1-7=1$

$$\Rightarrow \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$$

Solve $Ux = d$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$$

$x_3 = 1$

$\Rightarrow x_2 = \frac{7 - x_3}{3} = \frac{7-1}{3} = \frac{6}{3} = 2$

$\Rightarrow x_1 = \frac{1 + x_2 - x_3}{2} = \frac{1+2-1}{2} = \frac{2}{2} = 1$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore \quad x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$