CSCC37 TUT3 WEEK9
andrewyk.leung@mail.utoronto.ca
https://snoopysnipe.github.io/ta/c37f20/

Motivation behind GE:

- much easier to solve due to triangular forms of L and U
- More efficient to solve

$Ax = b_1$ $\qquad\qquad$ $A = LU$ $\quad O(n^3)$

$Ax = b_2$ $\qquad\qquad$ $\left.\begin{array}{l} LUx = b \\ Ld = b \\ Ux = d \\ n \cdot O(n^2) \end{array}\right\}$ $+ \; O(n^3)$

$\vdots$

$Ax = b_n$

since we can reuse the LU factorization as opposed to computing

$x = A^{-1} b_1$ $\qquad\qquad$ $FE = O(n^3)$

$x = A^{-1} b_2$ $\qquad\qquad$ $BS = O(n^2)$

$\vdots$ $\qquad\qquad$ $n \cdot O(n^3)$

$x = A^{-1} b_n$ $\qquad\qquad$ $O(n^4)$

Gaussian Elimination with (partial/row) Pivoting

① Compute $PA = LU$ factorization of matrix $A \in \mathbb{R}^{n \times n}$

② Use the factorization to solve system of linear equations $Ax = b$

1) Reduce A to upper triangular matrix form using $n-1$ Gauss transforms L_i interleaved with $n-1$ permutation matrices P_i

↳ L_i annihilates all elements on i-th column of A under a_{ii}

↳ Permute rows of A so that the largest element in magnitude on the i-th column below a_{ii} becomes the pivot (P_i interchanges rows i and j, $j > i$)

$\Rightarrow$ We have $L_{n-1} P_{n-1} L_{n-2} P_{n-2} \cdots L_2 P_2 L_1 P_1 A = U$

Note: $P_i P_i = I$ for any i

$$\Rightarrow L_{n-1} P_{n-1} L_{n-2} P_{n-2} \cdots L_2 P_2 L_1 \underbrace{P_2 P_2}_{I} P_1 A = U$$

Let $\tilde{L}_1 = P_2 L_1 P_2$

$\tilde{L}_1$ is just L_1 but with multipliers on rows i and j swapped where rows i and j are the same rows that P_2 swaps

ex: $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix}$

$$(P_2 L_1) P_2 = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & 0 & 1 \\ x_1 & 1 & 0 \end{bmatrix} P_2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_1 & 0 & 1 \end{bmatrix}$$

premultiplying swaps rows
postmultiplying swaps columns

$\Rightarrow$ continue until L_{n-2}

$\Rightarrow L_{n-1} \tilde{L}_{n-2} \dots \tilde{L}_2 \tilde{L}_1 P_{n-1} \dots P_3 P_2 P_1 A = U$

$\Rightarrow P_{n-1} \dots P_3 P_2 P_1 A = \tilde{L}_1^{-1} \tilde{L}_2^{-1} \dots \tilde{L}_{n-2}^{-1} L_{n-1} U$

$\Rightarrow PA = LU$

where $P = P_{n-1} P_{n-2} \dots P_2 P_1$

$L = \tilde{L}_1^{-1} \tilde{L}_2^{-1} \dots \tilde{L}_{n-2}^{-1} L_{n-1}$

<u>Recall:</u> $\tilde{L}_i$ is still a Gauss transform just with some multipliers interchanged

Compute L by toggling the signs (to calculate the individual inverses) and combining the elements under the diagonal (matrix multiplication)

2) Solve $Ax=b$:

$Ax=b \iff PAx=Pb \iff LUx=Pb$

i) solve $Ld=Pb$ for d (forward substitution)

ii) Solve $Ux=d$ for x (backward substitution)

<u>ex!</u> $Ax=b$ $\quad A=\begin{bmatrix} 2 & 6 & 6 \\ 3 & 5 & 12 \\ 6 & 6 & 12 \end{bmatrix}$ $\quad b=\begin{bmatrix} 20 \\ 25 \\ 30 \end{bmatrix}$

$P_1=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad P_1A=\begin{bmatrix} 6 & 6 & 12 \\ 3 & 5 & 12 \\ 2 & 6 & 6 \end{bmatrix}$

$L_1=\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix}$ $\quad L_1P_1A=\begin{bmatrix} 6 & 6 & 12 \\ 0 & 2 & 6 \\ 0 & 4 & 2 \end{bmatrix}$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P_2L_1P_1A = \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \qquad \boxed{L_2P_2L_1P_1A = \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U}$$

We have $L_2P_2L_1P_1A = U$

$\Leftrightarrow L_2P_2L_1P_2P_2P_1A = U$

$\Leftrightarrow L_2\tilde{L}_1P_2P_1A = U$

$\Leftrightarrow P_2P_1A = \tilde{L}_1^{-1}L_2^{-1}U$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}}$$

$$\boxed{P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

We have $Ax = b \Leftrightarrow PAx = Pb \Leftrightarrow LUx = Pb$

Let $Ux = d$

Solve $Ld = Pb$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \\ 25 \end{bmatrix}$$

$$d_1 = 30$$

$$\Rightarrow d_2 = 20 - \frac{1}{3} d_1 = 20 - 10 = 10$$

$$\Rightarrow d_3 = 25 - \frac{1}{2} d_2 - \frac{1}{2} d_1 = 25 - 5 - 15 = 5$$

$$\Rightarrow \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \\ 5 \end{bmatrix}$$

Solve $Ux = d$

$$\begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \\ 5 \end{bmatrix}$$

$$x_3 = 1$$

$$\Rightarrow x_2 = \frac{10 - 2x_3}{4} = \frac{10 - 2}{4} = 2$$

$$\Rightarrow x_1 = \frac{30 - 6x_2 - 12x_3}{6} = \frac{30 - 12 - 12}{6} = 1$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = x$$

ex: if A was 4×4:

$$L_3 P_3 L_2 P_2 L_1 P_1 A = U$$

$$L_3 P_3 L_2 P_2 L_1 P_2 P_2 P_1 A = U$$

$$L_3 P_3 L_2 \tilde{L}_1 P_2 P_1 A = U$$

$$L_3 \underbrace{P_3 L_2 P_3} \underbrace{P_3 \tilde{L}_1 P_3} P_3 P_2 P_1 A = U$$

$$L_3 \tilde{L}_2 \tilde{\tilde{L}}_1 P_3 P_2 P_1 A = U$$